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QUARTER-SYMMETRIC METRIC CONNECTION ON ALMOST KENMOTSU MANIFOLD WITH NULLITY DISTRIBUTION

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Abstract: The aim of the present paper is to characterized $(k,\mu)'$ almost Kenmotsu manifold admitting quarter-symmetric metric connection. Next we consider an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k,\mu)'$-nullity distribution satisfying certain curvature conditions with respect to quarter-symmetric metric connection.

Keywords and Phrases: Almost Kenmotsu manifold, quarter-symmetric metric connection, curvature tensor, CR-integrable, nullity distribution.

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1. Introduction

In 1969, S. Tanno [23] introduced the notion of almost contact metric manifolds whose automorphism groups attain the maximum dimensions and conclude that such manifolds can be classified into three classes. After that K. Kenmotsu in 1972, introduced a new type of almost contact metric manifolds which characterized the third case of Tanno's classification theorem and such manifolds were firstly named as Kenmotsu manifolds by Janssens and Vanhecke [15]. In 1981, Janssens and Vanhecke in [15] first introduced the notion of almost Kenmotsu manifolds which is a generalization of Kenmotsu manifolds. Almost Kenmotsu manifolds were recently studied by many authors as Dileo and Pastore [9, 10], Wang and Liu [24, 25, 26, 27], Ghosh [13], Deshmukh, De and Zhao [8].

In 1924, Friedmann and Schouten [11] introduced the idea of semi-symmetric connection on a differentiable manifold. In 1932, Hayden [14] introduced the idea of semi-symmetric metric connection on a Riemannian manifold. After that in 1970, K. Yano initiated systematic study of semi-symmetric connection in a Riemannian manifold. The study of a semi-symmetric connection was further developed by many other geometers. In 1975, S. Golab [12] introduced the notion of a quarter-symmetric linear connection in a differentiable manifold.

After S. Golab, S. C. Rastogi ([19], [20]) continued the systematic study of quarter-symmetric metric connection. Later in 1980, R. S. Mishra and S. N. Pandey [17] deduced some properties of the Riemannian, Kaehlerian and Sasakian manifolds that admits quarter-symmetric metric connection. In 1997, U. C. De and S. C. Biswas [3] studied a quarter-symmetric metric connection on a $SP-$Sasakian manifold. Also in 2008, Sular, Ozgur and De [22] studied a quarter-symmetric metric connection in a Kenmotsu manifold. The Quater-symmetric connection is also studied by A. Barman on $P-$Sasakian as well as on Kenmotsu manifold [5], [6].

Let ∇ be a linear connection in an $(2n+1)-$dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

A linear connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.1}$$

where η is a $1-$form and ϕ is a $(1,1)-$tensor field.

In particular, if $\phi(X) = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [11]. Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

If moreover, a quarter-symmetric connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0 \tag{1.2}$$

for all $X, Y, Z \in T(M)$, where $T(M)$ is the Lie algebra of vector fields of the manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection.

A Riemannian or a Semi-Riemannian manifold M is said to be Locally symmetric if its curvature tensor R satisfies $\nabla R = 0$, where ∇ is the Levi-Civita connection of the Riemannian manifold. As a generalization of locally symmetric manifold, many Geometers have considered Semi-symmetric manifold and in turn their generalizations. A Riemannian manifold is said to be Semi-symmetric [21] if its curvature tensor R satisfies

$$R(X, Y).R = 0, \quad X, Y \in TM$$

where $R(X, Y)$ is considered as a derivation and TM is the Lie algebra of vector fields of the manifold M.

Also a Riemannian manifold is said to be Ricci semi-symmetric if its curvature tensor R satisfies

$$R(X, Y).S = 0, \quad X, Y \in TM$$

where $R(X, Y)$ is considered as a field of linear operator acting on the Ricci tensor S of type $(0, 2)$. In 2005, E. Boeckx, P. Buecken and L. Vanhecke [4] introduced the notion of $\phi-$symmetry. Also in 2008, De and Sakar studied $\phi-$Ricci symmetric Sasakian manifolds [7]. An almost Kenmotsu manifold M^{2n+1} is said to be $\phi-$Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)Y) = 0,$$

for all $X, Y \in TM$. If X, Y are orthogonal to ξ, then manifold is said to be Locally $\phi-$Ricci symmetric.

The paper is organized as follows: After preliminaries in section 3 we recall some results in an almost Kenmotsu manifold with ξ belonging to $(k, \mu)'$-nullity distribution. In the next section we mention the relation between the Riemannian connection and the quarter-symmetric metric connection in that manifold. In section 5 we reduce the curvature tensor with respect to the quarter-symmetric metric connection and study some properties of Ricci tensor with respect to the said connection. In the next section we characterized Locally symmetric almost Kenmotsu manifolds with respect to the quarter-symmetric metric connection and we have proved that an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution with respect to quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to quarter-symmetric metric connection provided $k + 2 \neq 0$. In section 7 we consider the Ricci semi-symmetric almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution with respect to quarter-symmetric metric connection and conclude that there is no Ricci semi-symmetric almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution with respect to quarter-symmetric metric connection. Finally, we consider the Locally $\phi-$Ricci symmetric almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution with respect to quarter-symmetric metric connection and have the result: if an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution is locally $\phi-$Ricci symmetric with respect to quarter-symmetric metric connection, then the manifold is η Einstein provided $k + 2 \neq 0$.

2. Preliminaries

An $(2n+1)$-dimensional smooth differentiable manifold endowed with an almost contact structure (ϕ, ξ, η), if it admits a $(1, 1)$ tensor field ϕ, a characteristic vector field ξ, and a $1-$form η satisfying ([1], [2], [28]),

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \tag{2.1}$$

where I denotes the identity endomorphism on M. Also from the relation (2.1) it follows that $\eta(\xi) = 1, \quad \eta \circ \phi = 0$ and $rank(\phi) = 2n$.

If the manifold M with a structure (ϕ, ξ, η) admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields $X, Y \in TM$, then M is said to have an almost contact metric structure (ϕ, ξ, η, g). A smooth manifold endowed with an almost contact metric structure is called an almost contact metric manifold and it is denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. The fundamental $2-$form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields $X, Y \in TM$. Also, on the product manifold $M^{2n+1} \times \mathbb{R}$, an almost complex structure J is defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}), \tag{2.3}$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$. An almost contact structure is said to be normal if the above almost complex structure is integrable. According to Blair [1], the condition for an almost contact metric manifold being normal is equivalent to the vanishing of $(1, 2)-$type torsion tensor N_ϕ, defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion tensor of ϕ [1] defined as

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

An almost contact metric manifold with $d\eta = \Phi$ is called a contact metric manifold and a normal contact metric manifold is called a Sasakian manifold. On an almost contact metric manifold M^{2n+1} if both η and Φ are closed, then M is said to be an almost cosymplectic manifold and a normal almost cosymplectic manifold is said to be a cosymplectic manifold. According to Janssens and Vanhecke [15], an almost contact metric manifold satisfying $d\eta = 0$ and $d\Phi = d\eta \wedge \Phi$ is called an almost Kenmotsu manifold and a normal almost Kenmotsu manifold is said to be a Kenmotsu manifold.

Also an almost contact metric manifold M is said to be a Kenmotsu manifold [16] if the relation

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.4}$$

holds on M. For a Kenmotsu manifold we also have

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.5}$$

In [16] Kenmotsu showed: (a) that locally a Kenmotsu manifold is a Warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a non-zero constant; (b) that a Kenmotsu manifold of constant ϕ sectional curvature is a space of constant curvature -1 and so it is locally hyperbolic space.

Let M^{2n+1} be an almost Kenmotsu maifold. We consider now two tensor fields $l = R(.,\xi)\xi$ and $h = \frac{1}{2}\pounds_\xi \phi$ on M^{2n+1}, where R denotes the curvature tensor and $\pounds$ is the Lie differentiation. The tensor fields l and h are symmetric operators and satisfy the following relations:

$$h\xi = 0, \quad l\xi = 0, \quad tr(h) = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0, \tag{2.6}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi hX, \tag{2.7}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.8}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{2.9}$$

$$tr(l) = S(\xi,\xi) = -2n - tr(h^2), \tag{2.10}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.11}$$

for any vector fields $X, Y \in TM$, where S, Q, ∇ and tr denote the Ricci curvature tensor, the Ricci operator with respect to g, the Levi-Civita connection of g and the trace operator respectively. The $(1,1)-$type symmetric tensor field $h' = h \circ \phi$ is anti commuting with ϕ and $h'\xi = 0$. Also from ([9], [25]) we have

$$h = 0 \Leftrightarrow h' = 0, \quad h'^2 = (k+1)\phi^2 (h^2 = (k+1)\phi^2). \tag{2.12}$$

3. Almost Kenmotsu Manifold with ξ Belonging to $(k,\mu)'$-nullity Distribution

Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined as $\mathcal{D} = ker(\eta) = Im(\phi)$. Then on M^{2n+1} there exist an almost $CR-$structure $(\mathcal{D}, \phi_{\mathcal{D}})$, where $\phi_{\mathcal{D}}$ denotes the restriction of ϕ on distribution $\mathcal{D}$. In [9] Dileo and Pastore show that the almost $CR-$structure is integrable if and only if

$$[\phi X, \phi Y] - [X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] = 0,$$

for any $X, Y \in \mathcal{D}$, which is also equivalent to that the integral manifolds of $\mathcal{D}$ are Kählerian(i.e. $(1,1)-$type tensor field ϕ is $\eta-$parallel). If the integral manifold of $\mathcal{D}$ is Kählerian, then we say that M^{2n+1} is a $CR-$integrable almost Kenmotsu manifold [9].

In [2] Blair shows that any normal almost contact metric manifold is a $CR-$integrable manifold. This implies that every Kenmotsu manifold is always $CR-$integrable.

On an almost Kenmotsu manifold, if the characteristic vector field ξ satisfies

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{3.1}$$

for $X, Y \in TM$ with smooth functions k and μ, then M^{2n+1} is called a generalized $(k,\mu)-$ almost Kenmotsu manifold. Similarly, if on an almost Kenmotsu manifold, if the characteristic vector field ξ satisfies

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y), \tag{3.2}$$

for $X, Y \in TM$ with smooth functions k and μ and $h' = h \circ \phi$, then M^{2n+1} is called a generalized $(k,\mu)'-$ almost Kenmotsu manifold. If both k and μ in (3.1) and (3.2) are constant functions, then M^{2n+1} is said to be a (k,μ) or $(k,\mu)'-$ almost Kenmotsu manifold. According to [18] when $h = 0 (\Leftrightarrow h' = 0)$ then both generalized (k,μ) and generalized $(k,\mu)'-$ almost Kenmotsu manifold are $CR-$integrable.

Now we present some properties of $CR-$integrable almost kemotsu manifold.

Lemma 3.1. ([10], [18]) *An almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Kenmotsu manifold if and only if it is $CR-$integrable and h vanishes.*

Lemma 3.2. ([9]) *In an almost Kenmotsu manifold M^{2n+1} the distribution $\mathcal{D}$ has a Kählerian leaves if and only if*

$$(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX), \tag{3.3}$$

for any $X, Y \in TM$.

From Lemma 3.1 it follows that an almost Kenmotsu manifold with $CR-$ integrable structure is a Kenmotsu manifold if and only if $h = 0$. That is, a Kenmotsu manifold is an almost Kenmotsu manifold with $CR-$integrable structure, but the converse is not necessarily true.

Lemma 3.3. ([26]) *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n+1)-$dimensional almost Kenmotsu manifold with $CR-$integrable structure, then we have*

$$Q\phi - \phi Q = l\phi - \phi l + 4(n-1)h + (\eta \circ Q\phi) \otimes \xi - \eta \otimes \phi Q\xi, \tag{3.4}$$

$n \geq 1$.

According to [9], on a $(k, \mu)-$almost Kenmotsu manifold with $k \leq -1$, in [27] Wang have shown that $\mu = -2$, hence the following result is deduced by Wang directly from Dileo and Pastore [9]:

Lemma 3.4. (Lemma 3 of [27]) *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to $(k, \mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nI + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{3.5}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$. From the relation (3.5) we obtain

$$S(X, \phi Y) = -2ng(X, \phi Y) - 2ng(h'X, \phi Y) \tag{3.6}$$

and

$$S(\phi X, Y) = 2ng(X, \phi Y) - 2ng(h'X, \phi Y). \tag{3.7}$$

Combining above two equation we have

$$S(X, \phi Y) = S(\phi X, Y) + 4ng(X, \phi Y). \tag{3.8}$$

Similarly from (3.5) we get

$$S(X, hY) = S(hX, Y) + 4n(k+1)g(Y, \phi X). \tag{3.9}$$

Lemma 3.5. (Lemma 4.1 of [9]) *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, -2)'$-nullity distribution, then for any $X, Y \in TM$,*

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{3.10}$$

Contracting Y in (2.12) we have

$$S(X,\xi) = 2nk\eta(X). \tag{3.11}$$

Moreover in an almost Kenmotsu manifold with $(k,\mu)'-$nullity distribution we have

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(Y,h'X). \tag{3.12}$$

4. Relation between the Riemannian connection and the quarter - symmetric metric connection

If ∇ be a Riemannian connection of an almost contact metric manifold M and $\tilde{\nabla}$ be a quater-symmetric metric connection on M then, the relation between them is given by [12, 22]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{4.1}$$

In [22] Sular, Ozgur and De established the same relation between the Riemannian connection and the quarter-symmetric metric connection on a Kenmotsu manifold. Therefore equation (4.1) is the relation between the Riemannian connection and the quarter-symmetric metric connection on an almost Kenmotsu manifold.

5. Curvature tensor with respect to the quarter-symmetric metric connection

We define the curvature tensor of a $CR-$integrable almost Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{5.1}$$

In view of (4.1) we obtain

$$\begin{aligned}\tilde{R}(X,Y)Z =& R(X,Y)Z - (\nabla_X \eta)(Y)\phi Z + (\nabla_Y \eta)(X)\phi Z \\ &- \eta(Y)(\nabla_X \phi)Z + \eta(X)(\nabla_Y \phi)Z,\end{aligned} \tag{5.2}$$

which in view of (3.12) we have

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \eta(X)(\nabla_Y \phi)Z - \eta(Y)(\nabla_X \phi)Z. \tag{5.3}$$

Now applying (3.3) we get

$$\begin{aligned}\tilde{R}(X,Y)Z =& R(X,Y)Z + \{\eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z)\}\xi \\ &+ \{\eta(X)g(hY,Z) - \eta(Y)g(hX,Z)\}\xi \\ &- \{\eta(X)(\phi Y + hY) - \eta(Y)(\phi X + hX)\}\eta(Z).\end{aligned} \tag{5.4}$$

A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection ∇ is given by the relation (5.4). So from (5.4) and (2.11) we have

$$\tilde{R}(X,\xi)Y = R(X,\xi)Y + g(\phi X, Y)\xi - g(hX, Y) + \eta(Y)\phi X \tag{5.5}$$

and

$$\begin{aligned}\tilde{R}(X,Y)\xi =& \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) \\ &+ \eta(X)(h'Y + h'^2 Y) - \eta(Y)(h'X + h'^2 X) \\ &- \eta(X)(\phi Y + hY) + \eta(Y)(\phi X + hX).\end{aligned} \tag{5.6}$$

Using (2.12) we get

$$\begin{aligned}\tilde{R}(X,Y)\xi =& - k(\eta(X)Y - \eta(Y)X) + 2(\eta(X)h'Y - \eta(Y)h'X) \\ &- \eta(X)(\phi Y + hY) + \eta(Y)(\phi X + hX).\end{aligned} \tag{5.7}$$

In view of (5.7) we also obtain

$$\begin{aligned}\tilde{R}(\xi, Z)X =& - k(\eta(X)Z - g(X,Z)\xi) + 2(\eta(X)h'Z - g(h'X, Z)\xi) \\ &+ \eta(X)(\phi Z - hZ) + g(\phi X + hX, Z)\xi.\end{aligned} \tag{5.8}$$

Taking inner product of (5.4) with W we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) =& R(X,Y,Z,W) + \eta(X)\eta(W)\{g(\phi Y, Z) + g(hY, Z)\} \\ &- \eta(Y)\eta(W)\{g(\phi X, Z) + g(hX, Z)\} - \eta(X)\eta(Z)\{g(\phi Y, W) \\ &+ g(hY, W)\} + \eta(Y)\eta(Z)\{g(\phi X, W) + g(hX, W)\},\end{aligned} \tag{5.9}$$

where $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y,Z), W)$.

From (5.7) clearly

$$\tilde{R}(X,Y,Z,W) = -\tilde{R}(Y,X,Z,W), \tag{5.10}$$

$$\tilde{R}(X,Y,Z,W) = -\tilde{R}(X,Y,W,Z). \tag{5.11}$$

Combining above two relation we have

$$\tilde{R}(X,Y,Z,W) = \tilde{R}(Y,X,W,Z). \tag{5.12}$$

We also have

$$\begin{aligned}&\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y \\ &= 2\{\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi Z, X) + \eta(Z)g(\phi X, Y)\}.\end{aligned} \tag{5.13}$$

This is the first Bianchi identity for $\tilde{\nabla}$.

As ϕ is skew symmetric(i.e. $tr(\phi) = 0$), contracting (5.9) over X and W, we obtain

$$\tilde{S}(Y, Z) = S(Y, Z) + g(\phi Y, Z) + g(hY, Z) \tag{5.14}$$

and

$$\tilde{S}(Y, \xi) = S(Y, \xi) = 2nk\eta(Y), \tag{5.15}$$

where $\tilde{S}$ and S is the Ricci tensors of the connection $\tilde{\nabla}$ and ∇ respectively. Since in an almost Kenmotsu manifold the $(1,1)-$tensor field ϕ is skew symmetric, the Ricci tensor with respect to the quarter-symmetric metric connection is not symmetric and $\tilde{S}$ satisfies the relation

$$\tilde{S}(X, Y) = \tilde{S}(Y, X) + 2g(\phi X, Y). \tag{5.16}$$

Also from (5.14) we obtain

$$\tilde{Q}X = QX + \phi X + hX. \tag{5.17}$$

Again contracting (5.12), using (2.6) we have $\tilde{r} = r$, where $\tilde{r}$ and r are the scalar curvature of the connection $\tilde{\nabla}$ and ∇ respectively. So we have the following:

Proposition 5.1. *In an almost Kenmotsu manifold M with ξ belonging to $(k, \mu)'-$ nullity distribution with respect to quarter-symmetric metric connection then,*
(a) The curvature tensor $\tilde{R}$ is given by (5.9),
(b) The Ricci tensor $\tilde{S}$ is given by (5.14),
(c) The first Bianchi identity is given by (5.13),
(d) $\tilde{r} = r$,
(e) The Ricci tensor $\tilde{S}$ is not symmetric.

Since the Ricci tensor $\tilde{S}$ with respect to the quarter-symmetric metric connection is not symmetric, $\tilde{S}$ satisfies the following relations:

Replacing Y by hY in (5.14) we have

$$\tilde{S}(hY, Z) = S(hY, Z) + g(\phi hY, Z) + g(h^2Y, Z). \tag{5.18}$$

Using (2.12) and (3.9) from (5.18) we obtain

$$\begin{aligned}\tilde{S}(hY, Z) =& S(Y, hZ) - 4n(k + 1)g(Z, \phi Y) - g(h'Y, Z)\\ &- (k + 1)g(Y, Z) + (k + 1)\eta(Y)\eta(Z).\end{aligned} \tag{5.19}$$

Also using the relation (5.14) we have

$$\tilde{S}(hY, Z) = \tilde{S}(Y, hZ) - 2g(h'Y, Z) - 4n(k + 1)g(Z, \phi Y). \tag{5.20}$$

Similarly using (5.14) and (3.8) we have obtain the relation

$$\tilde{S}(\phi Y, Z) = S(\phi Z, Y) + 4ng(Z, \phi Y) - g(Y, Z) + g(h'Y, Z) + \eta(Y)\eta(Z). \quad (5.21)$$

Again using the relation (5.14) we have

$$\tilde{S}(\phi Y, Z) = \tilde{S}(Y, \phi Z) - 4ng(Y, \phi Z). \quad (5.22)$$

Replacing Z by hZ in (5.22) we have

$$\tilde{S}(\phi Y, hZ) = -\tilde{S}(Y, h'Z) + 4ng(Y, h'Z). \quad (5.23)$$

Combining the equations (5.20) and (5.23) we reduce the equation

$$\begin{aligned}\tilde{S}(h'Y, Z) = &- \tilde{S}(Y, h'Z) + 4ng(Y, h'Z) + 2g(hY, Z) \\ &+ 4n(k+1)\{g(Y, Z) - \eta(Y)\eta(Z)\}.\end{aligned} \quad (5.24)$$

6. Locally Symmetric almost Kenmotsu Manifolds with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity Distribution with respect to Quarter-symmetric Metric Connection

Assume that M is a Locally symmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$, then

$$(\tilde{\nabla}_X \tilde{R})(Y, Z)W = 0. \quad (6.1)$$

So by a suitable contraction of this equation we have

$$(\tilde{\nabla}_X \tilde{S})(Z, W) = \tilde{\nabla}_X \tilde{S}(Z, W) - \tilde{S}(\tilde{\nabla}_X Z, W) - \tilde{S}(Z, \tilde{\nabla}_X W) = 0. \quad (6.2)$$

Taking $W = \xi$ in the above equation we have

$$\tilde{\nabla}_X \tilde{S}(Z, \xi) - \tilde{S}(\tilde{\nabla}_X Z, \xi) - \tilde{S}(Z, \tilde{\nabla}_X \xi) = 0. \quad (6.3)$$

Now using (4.1), (2.7) and (3.11) we obtain from (6.3)

$$\tilde{S}(Z, X) + \tilde{S}(Z, h'X) = 2nk\{\eta(X)\eta(Z) + (\nabla_X \eta)Z\}. \quad (6.4)$$

Now if the characteristic vector field ξ belonging to $(k, \mu)'-$nullity distribution, then from (3.12) we have

$$\tilde{S}(Z, X) + \tilde{S}(Z, h'X) = 2nk\{g(X, Z) + g(h'X, Z)\}. \quad (6.5)$$

Substituting X by $h'X$ and using (2.12) and (5.15) we obtain

$$\tilde{S}(Z, h'X) - (k+1)\tilde{S}(Z, X) = 2nk\{g(h'X, Z) - (k+1)g(Z, X)\}. \tag{6.6}$$

Using (6.5) from above equation we obtain

$$(k+2)\{\tilde{S}(Z, X) - 2nkg(Z, X)\} = 0. \tag{6.7}$$

Hence we obtain the following:

Theorem 6.1. *Let M^{2n+1} be an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution with respect to quarter-symmetric metric connection $\tilde{\nabla}$, then the manifold is an Einstein manifold with respect to quarter-symmetric metric connection provided $k + 2 \neq 0$.*

7. Ricci Semi-symmetric almost Kenmotsu Manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity Distribution with respect to quarter-symmetric Metric Connection

Suppose that M be a Ricci semi-symmetric almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution with respect to quater-symmetric metric connection. So M^{2n+1} satisfies the condition

$$(\tilde{R}(X, Y).\tilde{S})(Z, W) = 0,$$

for any $X, Y, W, Z \in TM$.

Then we have

$$\tilde{S}(\tilde{R}(X, Y)Z, W) + \tilde{S}(Z, \tilde{R}(X, Y)W) = 0. \tag{7.1}$$

Putting $X = Z = \xi$ in (7.1), we have

$$\tilde{S}(\tilde{R}(\xi, Y)\xi, W) + \tilde{S}(\xi, \tilde{R}(\xi, Y)W) = 0. \tag{7.2}$$

Using (5.8) and (5.15) in (7.2), we have

$$\begin{aligned} k\tilde{S}(Y, W) - 2\tilde{S}(h'Y, W) - \tilde{S}(\phi Y, W) + \tilde{S}(hY, W) \\ = 2nk\{kg(Y, W) - 2g(h'Y, W) - g(\phi Y, W) + g(hY, W)\}. \end{aligned} \tag{7.3}$$

Interchanging Y and W in the above equation we obtain

$$\begin{aligned} k\tilde{S}(W, Y) - 2\tilde{S}(h'W, Y) - \tilde{S}(\phi W, Y) + \tilde{S}(hW, Y) \\ = 2nk\{kg(Y, W) - 2g(h'W, Y) - g(\phi W, Y) + g(hW, Y)\}. \end{aligned} \tag{7.4}$$

Now using (5.16), (5.24) and (5.20) the equation (7.4) yields

$$\begin{aligned} k\tilde{S}(Y,W)+&2\tilde{S}(W,h'Y) - \tilde{S}(W,\phi Y) + \tilde{S}(W,hY) \\ &= (2k-8n-2nk)g(\phi Y,W) + 2(4n-2nk+1)g(h'Y,W) \\ &\quad + (4+2nk)g(hY,W) + (8n(k+1)+2nk^2)g(Y,W) \\ &\quad - 8n(k+1)\eta(Y)\eta(W). \end{aligned} \tag{7.5}$$

Adding (7.3) and (7.5) and by the help of (5.16) we have

$$\begin{aligned} k\tilde{S}(Y,W)-&\tilde{S}(\phi Y,W) + \tilde{S}(hY,W) \\ &= (2nk+4)g(hY,W) + 4n(1-k)g(W,h'Y) \\ &\quad + (4nk+4n+2nk^2+1)g(Y,W) + (k-4n-2nk)g(\phi Y,W) \\ &\quad - (4n(k+1)+1)\eta(Y)\eta(W). \end{aligned} \tag{7.6}$$

Subtracting (7.6) from (7.3) we get

$$\begin{aligned} 2\tilde{S}(h'Y,W)-&4g(hY,W) - 4ng(W,h'Y) + (4n-k)g(\phi Y,W) \\ &-(4nk+4n+1)g(Y,W) + (4n(k+1)+1)\eta(Y)\eta(W) = 0. \end{aligned} \tag{7.7}$$

Replacing Y by $h'Y$ and using (2.12) in the above equation we obtain

$$\begin{aligned} 2(k+1)\tilde{S}(Y,W)-&(4nk+4n+1)g(h'Y,W) \\ &+4(k+1)g(\phi Y,W) + (4n-k)g(hY,W) \\ &= -4n(k+1)g(Y,W) + 4n(k+1)\eta(Y)\eta(W). \end{aligned} \tag{7.8}$$

Interchanging Y and W in the above equation and using (5.16) we obtain

$$\begin{aligned} 2(k+1)\tilde{S}(Y,W)-&(4nk+4n+1)g(h'Y,W) \\ &-8(k+1)g(\phi Y,W) + (4n-k)g(hY,W) \\ &= -4n(k+1)g(Y,W) + 4n(k+1)\eta(Y)\eta(W). \end{aligned} \tag{7.9}$$

Subtracting (7.8) from (7.9) we get

$$12(k+1)g(\phi Y,W) = 0. \tag{7.10}$$

Then the following two cases occur:
Case 1: $k+1=0$ implies $h'=0$, which contradicts the hypothesis $h' \neq 0$.
Case 2: $g(\phi Y,W)=0$ implies $h'=0$, which contradicts the hypothesis $h' \neq 0$.
Hence we have the following result.

Theorem 7.1. *There does not exist Ricci semi-symmetric almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution with respect to quater-symmetric metric connection.*

8. Locally $\phi-$Ricci Symmetric almost Kenmotsu Manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity Distribution with respect to Quarter-symmetric Metric Connection

Let us suppose that the manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution is Locally $\phi-$Ricci symmetric with respect to quater-symmetric metric connection. That is, the Ricci operator $\tilde{Q}$ satisfies the relation

$$\phi^2(\tilde{\nabla}_X \tilde{Q})Y = 0. \tag{8.1}$$

Now using (2.1) we have

$$-(\tilde{\nabla}_X \tilde{Q})Y + \eta((\tilde{\nabla}_X \tilde{Q})Y)\xi = 0, \tag{8.2}$$

which gives

$$\tilde{S}(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_X \tilde{Q}Y, Z) + \eta((\tilde{\nabla}_X \tilde{Q})Y)\eta(Z) = 0. \tag{8.3}$$

Substituting $Y = \xi$ in the above equation and using (5.18) and (2.7) we have

$$\tilde{S}(h'X, Z) + \tilde{S}(X, Z) + 2ng(h'X, Z) + 2ng(X, Z) - 4nk\eta(X)\eta(Z) = 0. \tag{8.4}$$

Replacing X by hX and using (2.12) we obtain

$$\begin{aligned} \tilde{S}(h'X, Z) &- (k+1)\tilde{S}(X, Z) + 2ng(h'X, Z) \\ &- 2n(k+1)g(X, Z) + 2n(k+1)^2\eta(X)\eta(Z) = 0. \end{aligned} \tag{8.5}$$

Using (8.4) in (8.5) we have

$$\tilde{S}(X, Z) = -2ng(X, Z) + \frac{2n}{k+2}\{k^2 + 4k + 1\}\eta(X)\eta(Z), \tag{8.6}$$

provided $k + 2 \neq 0$.

Theorem 8.1. *If an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to $(k, \mu)'$-nullity distribution is locally $\phi-$Ricci symmetric with respect to quarter-symmetric metric connection, then the manifold is $\eta-$Einstein with respect to quarter-symmetric metric connection provided $k + 2 \neq 0$.*

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